Diabatic gas flows

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The general equations of motion are developed for a compressible, inviscid flow in which a non-uniform distribution of heat transfer is applied to the fluid or a non-uniform generation of heat per unit volume occurs. In general, vorticity can be created if the cross-products of the temperature and entropy gradients are finite. If the temperature gradients in the flow are small (first-order), then a non-uniform heat addition across the stream will produce a second-order change in vorticity. For this type of flow, solutions are obtained for the variations of velocity and density that occur in a two-dimensional plane flow and an axially symmetric three-dimensional flow. A simple expression is also obtained for the streamline displacement caused by the non-uniform addition of heat.

1. Introduction

The coupling that exists between velocity, vorticity and gradients of stagnation enthalpy and entropy is well known. Tsien (1958) and Vazsonyi (1945) have reviewed the equations of motion for gas flows in which temperature and entropy gradients are present. The work described in this paper deals with the flow which is produced when a non-uniform distribution of heat transfer is applied to a gaseous stream or a non-uniform generation of heat per unit volume occurs. In practice such flows may take place in heat exchangers and nuclear reactors when a local hot spot develops, or in the flow through a flame front in which energy is released non-uniformly across the stream.

The general equations of motion for an inviscid, compressible fluid with vanishing thermal conductivity are first developed, a term allowing for the heat addition being included in the energy equation. The equations for a two-dimensional diabatic flow are next considered, and the equations for the velocity, vorticity and density perturbations are obtained. For the special case in which the temperature gradients are small, two analytical methods of solution and one numerical method are given for subsonic flows. A simple example is given to illustrate two of these methods and to allow a comparison to be made between the two solutions. A brief discussion of an axially symmetric diabatic gas flow then follows, and an example of such a flow is given.

2. The general equations of motion for a diabatic gas flow

The momentum equation for an inviscid flow is

$$-\frac{1}{\rho}\nabla p = (\mathbf{q}.\nabla)\,\mathbf{q}+\frac{\partial \mathbf{q}}{\partial t}, \tag{1}$$

where p is the pressure, ρ the density and $\mathbf{q}$ the velocity vector. Alternatively, using the identity

$$\nabla(\tfrac{1}{2}\mathbf{q}^2) = (\mathbf{q}\,.\nabla)\,\mathbf{q}+\mathbf{q}\times\mathbf{w},$$

where $\mathbf{w} = \nabla\times\mathbf{q}$ is the vorticity, equation (1) may be written as

$$\frac{1}{\rho}\nabla p+\nabla(\tfrac{1}{2}\mathbf{q}^2) = \mathbf{q}\times\mathbf{w} \tag{2}$$

for a steady flow.

The stagnation enthalpy h_0 is defined in terms of the enthalpy h and the velocity magnitude q by the equation

$$h_0 = h+\tfrac{1}{2}q^2, \tag{3}$$

so that

$$\nabla h_0 = \nabla h+\nabla(\tfrac{1}{2}q^2). \tag{4}$$

From the Gibbs equation relating the properties of the fluid,

$$T\nabla s = \nabla h-\frac{1}{\rho}\nabla p, \tag{5}$$

Crocco's equations is obtained as

$$\nabla h_0 - T\nabla s = \mathbf{q}\times\mathbf{w}, \tag{6}$$

where T is the temperature and s the entropy of the fluid.

If the heat addition to the fluid is $\dot{Q}$ per unit volume per unit time, then the steady-flow energy equation is

$$\dot{Q} = \nabla\,.\,(\rho\mathbf{q}h_0). \tag{7}$$

Since the flow is steady, the equation of continuity is

$$\nabla\,.\,(\rho\mathbf{q}) = 0, \tag{8}$$

so that (7) may be expressed as

$$\dot{Q} = \rho\mathbf{q}\,.\nabla h_0. \tag{9}$$

By taking the curl of (6), an equation corresponding to the Helmholtz equation is obtained:

$$\nabla T\times\nabla s = (\mathbf{q}\,.\nabla)\,\mathbf{w}-(\mathbf{w}\,.\nabla)\,\mathbf{q}+\mathbf{w}(\nabla\,.\,\mathbf{q}). \tag{10}$$

For barotropic flows (e.g. isentropic or isothermal flows), $\nabla T\times\nabla s$ is zero, which is apparent from taking the curl of (5). The usual form of the Helmholtz equation for a barotropic flow then follows from (10).

If the fluid is a perfect gas with $p = R\rho T$ and c_p, the specific heat at constant pressure, a constant, then the steady-flow energy equation is

$$\frac{\dot{Q}}{\rho c_p} = \mathbf{q}\,.\nabla\left(T+\frac{q^2}{2c_p}\right). \tag{11}$$

From equations (1) and (8), the first term on the right-hand side of (11) is

$$\begin{aligned}\mathbf{q}\,.\nabla T &= \frac{\mathbf{q}}{R}.\nabla\left(\frac{p}{\rho}\right)\\ &= \frac{\mathbf{q}}{R}.\left(\frac{1}{\rho}\nabla p-\frac{p}{\rho^2}\nabla\rho\right)\\ &= \frac{p}{\rho R}\nabla\,.\,\mathbf{q}-\frac{\mathbf{q}}{R}.\,(\mathbf{q}\,.\nabla)\,\mathbf{q}.\end{aligned} \tag{12}$$

Using the identity given above (2) along with (12), equation (11) may be written as

$$\left(\frac{k-1}{k}\right)\frac{\dot{Q}}{p} = \nabla.\mathbf{q} - \frac{\mathbf{q}}{kRT}.(\mathbf{q}.\nabla)\,\mathbf{q}, \tag{13}$$

where k is the ratio of the specific heats of the gas. This equation has proved to be the most useful form of the steady-flow energy equation for the purpose of this paper.

3. Two-dimensional flows with heat addition

When a perfect gas flows along a parallel-sided duct, the presence of a non-uniform addition of heat can produce a region in which the streamlines are not parallel to the walls of the duct. By using a perturbation method for a heat addition with only a small degree of non-uniformity, it is possible to demonstrate some of the characteristics of this type of flow. As shown in figure 1, the x-axis is taken along the duct wall, the y-axis lies across the duct and there is no flow normal to the (x, y)-plane. The addition of heat is non-uniformly distributed along and across the duct.

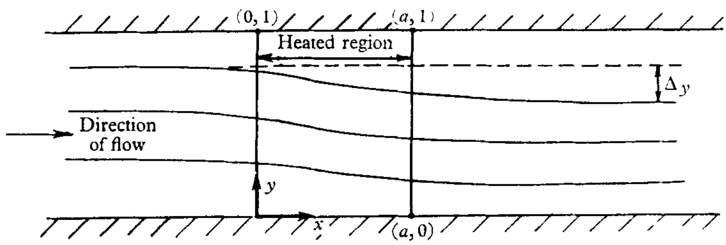

FIGURE 1. Two-dimensional diabatic gas flow.

In a two-dimensional flow where u and v are the x- and y-components of velocity respectively, and η is the only component of vorticity, in a direction normal to the (x, y)-plane, the equations of momentum, continuity and energy are

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}, \tag{14}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}, \tag{15}$$

$$0 = \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v), \tag{16}$$

$$\left(\frac{k-1}{k}\right)\frac{\dot{Q}}{p} = \left[1 - \frac{u^2}{kRT}\right]\frac{\partial u}{\partial x} + \left[1 - \frac{v^2}{kRT}\right]\frac{\partial v}{\partial y} - \frac{uv}{kRT}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]. \tag{17}$$

For this two-dimensional flow, equation (10) becomes

$$\frac{\partial T}{\partial x}\frac{\partial s}{\partial y} - \frac{\partial T}{\partial y}\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}(u\eta) + \frac{\partial}{\partial y}(v\eta), \tag{18}$$

which shows the possibility of producing changes in vorticity by the presence of temperature and entropy gradients.

It is assumed that the Mach number of the flow is less than unity, and that far upstream of the region of heat addition the gas has a uniform velocity $\bar{U}$ along the duct and a Mach number $\bar{M}$. The properties of the gas far upstream are also assumed to be uniform across the duct and are used to make the above equations non-dimensional by writing

$$\left.\begin{aligned} x = bx', \quad y = by', \quad u = \bar{U}u', \quad v = \bar{U}v', \\ \eta = \frac{\bar{U}}{b}\eta', \quad p = \bar{P}p', \quad \rho = \bar{D}\rho', \quad T = \bar{T}\theta, \\ \left(\frac{k-1}{k}\right)\dot{Q} = \frac{\bar{U}\bar{P}}{b}F(x', y'), \end{aligned}\right\} \tag{19}$$

where $\bar{P}$, $\bar{T}$ and $\bar{D}$ are the pressure, temperature and density far upstream and b is the width of the duct. The non-dimensional forms of equations (14) to (17) are then

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = k\bar{M}^2\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right], \tag{20}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = k\bar{M}^2\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right]. \tag{21}$$

$$0 = \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v), \tag{22}$$

$$\frac{1}{p}F(x,y) = \left[1 - \bar{M}^2\frac{u^2}{\theta}\right]\frac{\partial u}{\partial x} + \left[1 - \bar{M}^2\frac{v^2}{\theta}\right]\frac{\partial v}{\partial y} - \bar{M}^2\frac{uv}{\theta}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right], \tag{23}$$

where, for convenience, the primes have been omitted.

If the heat addition function $F(x, y)$ can be expressed as

$$F(x,y) = F_0(x) + \epsilon f(x,y), \tag{24}$$

where $\epsilon \ll 1$ and $F_0(x)$ and $f(x, y)$ are of the same order of magnitude, then a perturbation method may be applied to the problem. The function $F_0(x)$ represents a heat addition which can vary along the duct, but is uniform across the duct. The second term, $\epsilon f(x, y)$, indicates the presence of a small non-uniform heat addition which can vary both along and across the duct. It is this non-uniform heat addition that produces the displacement of the streamlines and the creation of vorticity.

It is assumed that the variables u, v, η, p, ρ and θ are analytic in ϵ and may be expressed as power series in ϵ, the general form being

$$\left.\begin{aligned} u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots, \\ v &= \epsilon v_1 + \epsilon^2 v_2 + \ldots, \end{aligned}\right\} \tag{25}$$

etc.

If equations (24) and (25) are substituted in equations (20) to (23), then, equating the zero powers of ϵ, we get

$$\left.\begin{aligned} -\frac{1}{\rho_0}\frac{\partial p_0}{\partial x} &= k\bar{M}^2 u_0\frac{\partial u_0}{\partial x}, \\ -\frac{1}{\rho_0}\frac{\partial p_0}{\partial y} &= 0, \\ 0 &= \frac{\partial}{\partial x}(\rho_0 u_0), \\ \frac{1}{p_0}F_0(x) &= \left[1 - \bar{M}^2\frac{u_0^2}{\theta_0}\right]\frac{\partial u_0}{\partial x}. \end{aligned}\right\} \tag{26}$$

These are the equations for a one-dimensional Rayleigh-line process, the addition of heat to a one-dimensional steady flow. The solution is well known; the velocity and gas state vary along the duct, but there is no variation across the flow.

By equating the first powers of ϵ in equations (20) to (23), the equations relating the first-order perturbations in velocity, pressure, density and temperature are obtained:

$$\frac{\rho_1}{\rho_0^2}\frac{dp_0}{dx}-\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} = k\bar{M}^2\left[u_0\frac{\partial u_1}{\partial x}+u_1\frac{du_0}{dx}\right], \tag{27}$$

$$-\frac{1}{\rho_0}\frac{\partial p_1}{\partial y} = k\bar{M}^2 u_0\frac{\partial v_1}{\partial x}, \tag{28}$$

$$0 = \frac{\partial}{\partial x}(u_0\rho_1+u_1\rho_0)+\frac{\partial}{\partial y}(\rho_0 v_1), \tag{29}$$

$$\frac{f(x,y)}{p_0}-\frac{p_1}{p_0^2}F_0(x) = \left[1-\bar{M}^2\frac{u_0^2}{\theta_0}\right]\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}-\bar{M}^2\frac{du_0}{dx}\left[\frac{2u_0u_1}{\theta_0}-\frac{\theta_1 u_0^2}{\theta_0^2}\right]. \tag{30}$$

The solution of these equations gives the perturbations of velocity and gas state which are superposed on the one-dimensional Rayleigh-line process.

If p_1 and ρ_1 are eliminated from the momentum and continuity equations ((27) to (29)), then an equation connecting the two perturbation velocities is obtained:

$$u_0\frac{\partial}{\partial x}\left[\frac{1}{du_0/dx}\left(\frac{\partial^2 v_1}{\partial x^2}-\frac{\partial^2 u_1}{\partial x\,\partial y}\right)\right]+\frac{\partial^2 v_1}{\partial y^2} = 0. \tag{31}$$

Introducing the vorticity and the uniform heat addition, this equation may be written as

$$u_0\frac{\partial}{\partial x}\left[\frac{p_0}{F_0(x)}\left(1-\bar{M}^2\frac{u_0^2}{\theta_0}\right)\frac{\partial \eta_1}{\partial x}\right]+\frac{\partial^2 v_1}{\partial y^2} = 0. \tag{32}$$

A further equation relating the two perturbation velocities may be derived by using the equation of state for the gas to eliminate θ_1 from (30), and then differentiating with respect to y to obtain

$$\frac{1}{p_0}\frac{\partial f}{\partial y}(x,y)+\bar{M}^2\frac{u_0}{\theta_0}\frac{du_0}{dx}\left[k\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{\partial y}\right] = \frac{\partial^2 u_1}{\partial x\,\partial y}+\frac{\partial^2 v_1}{\partial y^2}-\bar{M}^2\frac{u_0^2}{\theta_0}\frac{\partial^2 v_1}{\partial x^2}. \tag{33}$$

The two velocities u_1 and v_1 are therefore determined by (31) and (33), along with the appropriate boundary conditions.

Two of the boundary conditions are those imposed by the duct walls, namely $v = 0$ on $y = 0$ and $y = 1$. It then follows from (32) that in the presence of a heat addition for which $F_0(x)$ is not zero, the vorticity perturbation η_1 can only be zero if the transverse velocity v_1 is also zero throughout the flow. This is the case of the one-dimensional Rayleigh-line process where the flow remains parallel to the walls of the duct and $\epsilon f(x,y) = 0$. When $F_0(x) = 0$ and only the small non-uniform heat addition is present, the change of vorticity may be shown to be a second- or higher-order perturbation. This case is examined in more detail in the remainder of this paper. In general, if neither $F_0(x)$ nor $\epsilon f(x,y)$ are zero, then there will be a first-order perturbation in vorticity.

4. Two-dimensional flows with $F_0(x) = 0$

No solution has yet been obtained for the general case when both $F_0(x)$ and $\epsilon f(x, y)$ are present. However, it is possible to observe some of the features of two-dimensional diabatic gas flows by considering the special case of $F_0(x) = 0$. The heat addition then consists of the small non-uniform heat addition alone and the temperature gradients are therefore small. For this type of heat addition, a general solution for the first-order velocity perturbations has been derived in the form of an integral, and for a particular distribution of heat addition both a Fourier series solution and a numerical solution have also been obtained.

In the absence of the uniform heat addition $F_0(x)$, the solution of equations (26) for the one-dimensional Rayleigh-line process is that u_0, p_0, ρ_0 and θ_0 are each constant and equal to unity, their value far upstream. The Mach number of the flow is therefore constant and equal to $\bar{M}$, the value far upstream. The first-order perturbations of velocity, pressure and density are then given by

$$-\frac{\partial p_1}{\partial x} = k\bar{M}^2 \frac{\partial u_1}{\partial x}, \tag{34}$$

$$-\frac{\partial p_1}{\partial y} = k\bar{M}^2 \frac{\partial v_1}{\partial x}, \tag{35}$$

$$0 = \frac{\partial}{\partial x}(u_1 + \rho_1) + \frac{\partial v_1}{\partial y}, \tag{36}$$

$$f(x, y) = [1 - \bar{M}^2] \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}, \tag{37}$$

which correspond to (27) to (30) of the general analysis.

The elimination of p_1 between (34) and (35) gives

$$\frac{\partial \eta_1}{\partial x} = 0, \tag{38}$$

and since the flow far upstream is uniform, the first-order vorticity perturbation is zero throughout the entire flow. It is therefore possible to define a perturbation velocity potential $\phi(x, y)$ such that

$$u_1 = \frac{\partial \phi}{\partial x}, \quad v_1 = \frac{\partial \phi}{\partial y}. \tag{39}$$

Equation (37) may then be written as

$$[1 - \bar{M}^2] \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y). \tag{40}$$

If the heat addition is confined to a finite length of the duct, say $x = 0$ to $x = a$ in the non-dimensional system, then the boundary conditions for (40) are

$$\left.\begin{aligned} &\frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad y = 0 \text{ and } y = 1, \\ &\frac{\partial \phi}{\partial x} \to 0, \quad \frac{\partial \phi}{\partial y} \to 0 \quad \text{as} \quad x \to -\infty, \\ &\frac{\partial \phi}{\partial x} \to \alpha^2 \int_0^a \int_0^1 f(x, y)\, dy\, dx \quad \text{as} \quad x \to +\infty, \end{aligned}\right\} \tag{41}$$

where $\alpha^2 = 1/(1-\bar{M}^2)$. The second pair of conditions are necessary to leave the flow far upstream unperturbed and the final boundary condition is obtained from the other conditions by applying Green's Theorem to (40).

Integral solution

The solution of (40) for a unit heat source situated at the point (x_0, y_0) is the Green function $G(x, y; x_0, y_0)$, for the problem. It can be shown that the perturbation velocity potential may then be expressed as

$$\phi(x,y) = \int_0^a \int_0^1 f(x_0, y_0)\, G(x_0, y_0; x, y)\, dy_0\, dx_0 + C, \tag{42}$$

where C is a constant.

The Green function satisfies the equation

$$[1-\bar{M}^2]\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \delta(x-x_0)\,\delta(y-y_0), \tag{43}$$

and has the boundary conditions

$$\left.\begin{aligned} &\frac{\partial G}{\partial y} = 0 \quad \text{on} \quad y = 0 \text{ and } y = 1, \\ &\frac{\partial G}{\partial x} \to 0, \quad \frac{\partial G}{\partial y} \to 0 \quad \text{as} \quad x \to -\infty, \\ &\frac{\partial G}{\partial x} \to \alpha^2 \quad \text{as} \quad x \to +\infty. \end{aligned}\right\} \tag{44}$$

The solution can be obtained by the method of images and takes the form

$$\begin{aligned} G(x_0, y_0; x, y) = \frac{\alpha}{4\pi} \log \{4[\cosh \pi\alpha(x-x_0) - \cos \pi(y-y_0)] \\ \times [\cosh \pi\alpha(x-x_0) - \cos \pi(y+y_0)]\} + \tfrac{1}{2}\alpha^2(x+x_0) + \text{const.} \end{aligned} \tag{45}$$

If this expression is substituted in (42), then the two perturbation velocities u_1 and v_1 may be obtained by forming the gradient of $\phi(x, y)$. In practice this method is difficult to apply and an approximate solution is more suitable for estimating the flow pattern.

Fourier series solution

A solution for the perturbation velocity potential has been derived for the case when the heat addition function is given by $f(x, y) = y$. A numerical solution has also been obtained for the transverse velocity v_1 that is produced by this form of heat addition, and this may be compared with the value which is given by the Fourier series solution.

The equation governing the flow outside of the heated region is

$$[1-\bar{M}^2]\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{46}$$

and, within the region of heat addition,

$$[1-\bar{M}^2]\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = y. \tag{47}$$

In order to provide the necessary boundary conditions for these two equations, it is assumed that at the two boundaries of the heated region, $x = 0$ and $x = a$, there is no discontinuity in the gradient of the perturbation velocity potential. The physical interpretation of this requirement is that the two perturbation velocities, u_1 and v_1, are continuous throughout the flow.

If $m = 2n-1$, then the solution of (46) and (47) is that, upstream,

$$\phi(x,y) = \frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{e^{m\pi\alpha x}}{m^4}[1-e^{-m\pi\alpha a}]\cos(m\pi y), \tag{48}$$

in the heated region,

$$\phi(x,y) = \frac{\alpha^2 x^2}{4}+\frac{y^2(2y-3)}{12}-\frac{2}{\pi^4}\sum_{n=1}^{\infty}\left[\frac{e^{-m\pi\alpha x}+e^{m\pi\alpha(x-a)}}{m^4}\right]\cos(m\pi y), \tag{49}$$

and downstream,

$$\phi(x,y) = \frac{\alpha^2 a x}{2}-\frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{e^{-m\pi\alpha x}}{m^4}[1-e^{m\pi\alpha a}]\cos(m\pi y). \tag{50}$$

The two velocities u_1 and v_1 are then obtained by forming the gradient of $\phi(x,y)$, and it is clear that since the coefficients of the Fourier series for these velocities decrease very rapidly, a good approximation can be obtained by considering only the first terms of the series. The solution also shows that the production of a transverse velocity v_1 is a local effect which does not extend far outside of the region of heat addition.

Numerical solution

The displacement of the streamlines is determined by the transverse velocity v_1 and to the first order in ϵ, the direction of the flow is given by

$$\frac{dy}{dx} = \frac{\epsilon v_1}{u_0} = \epsilon v_1. \tag{51}$$

When considering the characteristics of a diabatic flow, it is useful to know the streamline pattern and the numerical solution was chosen so that the transverse velocity would be calculated directly.

The equation governing the velocity perturbation v_1 is obtained by differentiating (37) with respect to y and eliminating u_1. With $f(x,y) = y$, this equation is

$$\frac{\partial^2 v_1}{\partial z^2}+\frac{\partial^2 v_1}{\partial y^2} = 1 \quad \text{within the heated region,}$$

and

$$\frac{\partial^2 v_1}{\partial z^2}+\frac{\partial^2 v_1}{\partial y^2} = 0 \quad \text{elsewhere,}$$

where $z = \alpha x$. The boundary conditions for these equations are

$$v_1 = 0 \quad \text{on} \quad y = 0 \text{ and } y = 1,$$

and

$$v_1 \to 0 \quad \text{as} \quad z \to \pm\infty.$$

Since the production of a transverse velocity is a local effect, the boundary conditions, as z tends towards infinity, are applied at finite distances upstream and downstream of the heated region which extends from $z = 0$ to $z = 1$. As in the Fourier series solution, it is assumed that the perturbation velocity v_1 is continuous throughout the region of flow. The mesh size for the difference equation was chosen as $l = \frac{1}{6}$, and the boundary conditions were taken as

$$v_1 = 0 \quad \text{on} \quad y = 0 \text{ and } y = 1, \quad \text{on} \quad z = -2{\cdot}41\dot{6}, \quad \text{and on} \quad z = 3{\cdot}41\dot{6}. \tag{52}$$

If $W^n_{i,j}$ is the nth approximation to $v_1(il, jl)$, then within the heated region the iterative process for solving the difference equation for v_1 is

$$W^{n+1}_{i,j} = W^n_{i,j} + \tfrac{1}{4}[W^n_{i+1,j} + W^n_{i,j+1} + W^{n+1}_{i-1,j} + W^{n+1}_{i,j-1} - 4W^n_{i,j} - \tfrac{1}{36}], \tag{53}$$

and, outside the heated region,

$$W^{n+1}_{i,j} = W^n_{i,j} + \tfrac{1}{4}[W^n_{i+1,j} + W^n_{i,j+1} + W^{n+1}_{i-1,j} + W^{n+1}_{i,j-1} - 4W^n_{i,j}]. \tag{54}$$

z	y	W	v_F
0·4167	0·0000	0·0000	0·0000
0·4167	0·1667	−0·0555	−0·0555
0·4167	0·3333	−0·0871	−0·0870
0·4167	0·5000	−0·0973	−0·0971
0·4167	0·6667	−0·0871	−0·0870
0·4167	0·8333	−0·0555	−0·0555
0·4167	1·0000	0·0000	0·0000

W, numerical solution for v_1. v_F, first term of the Fourier series solution.

TABLE 1

These two equations, along with the boundary conditions given in (52), were solved with the aid of a digital computer, and table 1 shows that there is very close agreement between the values obtained by this method and the values obtained from the first term of the Fourier series solution for v_1 with $a\alpha = 1$.

5. Density perturbations with $F_0(x) = 0$

The non-uniform addition of heat to a perfect gas flow produces both an overall change in the mean density and a density gradient across the duct. From equations (36) and (37), we have

$$\frac{\partial \rho_1}{\partial x} = -f(x,y) - \bar{M}^2 \frac{\partial u_1}{\partial x}. \tag{55}$$

Since both ρ_1 and u_1 are zero far upstream, this equation may be integrated to give

$$\rho_1 = -\int_{-\infty}^{x} f(x, y)\, dx - \bar{M}^2 u_1. \tag{56}$$

If the heat addition is confined to the region $0 \leqslant x \leqslant a$, then, for $x > a$,

$$\rho_1 = -\int_0^a f(x,y)\, dx - \bar{M}^2 u_1,$$

and the density perturbation far downstream is

$$\rho_1 = -\left\{\int_0^a f(x,y)\, dx + \alpha^2 \bar{M}^2 \int_0^a \int_0^1 f(x,y)\, dy\, dx\right\}. \tag{57}$$

This equation shows that the density gradient across the duct is not a local effect like the transverse velocity, but is a feature which is retained in the flow far downstream of the heated region.

6. Streamline displacement with $F_0(x) = 0$

The presence of a transverse velocity causes a displacement of the streamlines, and it has been shown that the flow direction is given by $dy/dx = \epsilon v_1$. The streamline displacement Δy from far upstream to far downstream is then

$$\Delta y = \epsilon \int_{-\infty}^{\infty} v_1 \, dx, \tag{58}$$

where the integration is performed along the streamline. To the first order in ϵ, this integration may be approximated by an integration parallel to the x-axis. Thus, introducing the perturbation velocity potential, we have

$$\Delta y = \epsilon \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial y} dx. \tag{59}$$

This integral can be evaluated by applying Green's theorem to (40) for the region bounded by the streamline and the duct wall. The streamline displacement can then be expressed as

$$\Delta y = \epsilon \left\{ \int_0^a \int_0^y f(x,y) \, dy \, dx - y \int_0^a \int_0^1 f(x,y) \, dy \, dx \right\}. \tag{60}$$

The same result may also be obtained by considering the continuity of the fluid between the streamline and the duct wall and using the equation for the density perturbation.

Equation (60) is very useful for visualizing the type of flow pattern that is produced by simple heat addition functions such as linear, sinusoidal and step distributions. For the linear distribution $f(x,y) = y$, the displacement is given by

$$\Delta y = \tfrac{1}{2} \epsilon a y (y-1), \tag{61}$$

and when $y = 0{\cdot}5$, $\Delta y = -0{\cdot}125\epsilon a$. The displacement can also be estimated from the numerical solution by using a summation in place of the integration in (59). The value obtained by this method is $\Delta y = -0{\cdot}12495\epsilon a$ for $y = 0{\cdot}5$. The very close agreement between these two values and also between those given in table 1 confirms the suitability of the mesh size and the location of the boundary conditions in the numerical solution.

7. Three-dimensional diabatic gas flows with axial symmetry

A similar analysis can be made for the non-uniform addition of heat to an axially symmetric perfect gas flow. The flow, as shown in figure 2, is assumed to take place in a circular duct and the velocity and gas state far upstream of the region of heat addition are assumed to be uniform across the duct. The non-dimensional heating function is expressed as

$$F(x,r) = F_0(x) + \epsilon f(x,r),$$

and the velocity and gas state are written in the form of perturbation series. It can then be shown that the first-order perturbation of vorticity can be zero only if $F_0(x)$ or $\epsilon f(x,r)$ is zero. This corresponds to the result which was deduced from (32) for the two-dimensional flow.

When $F_0(x) = 0$, the first-order vorticity perturbation is zero, and it is possible to define a perturbation velocity potential $\phi(x, r)$ such that

$$u_1 = \frac{\partial \phi}{\partial x}, \quad v_1 = \frac{\partial \phi}{\partial r}, \tag{62}$$

where r is the non-dimensional radius and v_1 is the radial velocity perturbation. The velocity potential $\phi(x, r)$ then satisfies the equation

$$[1-\bar{M}^2]\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}=f(x,r), \tag{63}$$

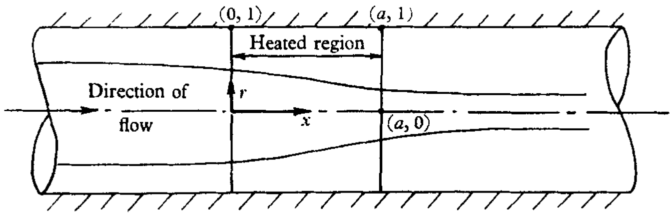

FIGURE 2. Three-dimensional diabatic gas flow with axial symmetry.

which corresponds to (40) of the two-dimensional analysis. The boundary conditions for (63) are

$$\left.\begin{aligned} &\frac{\partial \phi}{\partial r} = 0 \quad \text{on} \quad r = 0 \text{ and } r = 1, \\ &\frac{\partial \phi}{\partial x} \to 0, \quad \frac{\partial \phi}{\partial r} \to 0 \quad \text{as} \quad x \to -\infty, \\ &\frac{\partial \phi}{\partial x} \to 2\alpha^2 \int_0^a \int_0^1 r f(x,r)\,dr\,dx \quad \text{as} \quad x \to +\infty, \end{aligned}\right\} \tag{64}$$

where it is assumed that the heated region extends from $x = 0$ to $x = a$.

For the parabolic distribution of heat addition defined by $f(x, r) = r^2$, a solution for $\phi(x, r)$ has been obtained in terms of zero-order Bessel functions. The flow upstream of the heated region is given by

$$\phi(x,r) = -2\sum_{n=1}^{\infty}\frac{e^{k_n\alpha x}}{k_n^4}[1-e^{-k_n\alpha a}]\frac{J_0(k_n r)}{J_0(k_n)}, \tag{65}$$

within the heated region by

$$\phi(x,r) = \frac{\alpha^2 x^2}{4}+\frac{r^2(r^2-2)}{16}+2\sum_{n=1}^{\infty}\left[\frac{e^{-k_n\alpha x}+e^{k_n\alpha(x-a)}}{k_n^4}\right]\frac{J_0(k_n r)}{J_0(k_n)}, \tag{66}$$

and downstream by

$$\phi(x,r) = \frac{\alpha^2 a x}{2}+2\sum_{n=1}^{\infty}\frac{e^{-k_n\alpha x}}{k_n^4}[1-e^{k_n\alpha a}]\frac{J_0(k_n r)}{J_0(k_n)}, \tag{67}$$

where
$$J_1(k_n) = 0. \tag{68}$$

As with the Fourier series solution, a good approximation may be obtained by taking only the first few terms of the series.

The method for estimating the streamline displacement when $F_0(x) = 0$ can be extended to the three-dimensional axially symmetric flow to give

$$\Delta r = \epsilon\left\{\frac{1}{r}\int_0^a \int_0^r rf(x,r)\,dr\,dx - r\int_0^a \int_0^1 rf(x,r)\,dr\,dx\right\}. \tag{69}$$

For the particular form of heat addition considered above, the displacement is $\Delta r = \frac{1}{4}\epsilon ar(r^2-1)$, which has a maximum value of $\Delta r = -0{\cdot}104\epsilon a$ when $r = 0{\cdot}577$. The general form of this flow is as shown in figure 2.

8. Discussion

For compressible gas flows, vorticity can be created if the cross-products of the temperature and entropy gradients are finite. In particular, for the flow of a perfect gas in a parallel walled duct, it has been shown that except when the temperature gradients are small, a non-uniform heat addition will produce a change in vorticity. A similar result can also be obtained for a three-dimensional flow in a circular duct with axial symmetry. In both cases it has been assumed that far upstream of the heated region, the velocity and gas state are uniform across the duct.

The two solutions given in this paper are for the special case when the temperature gradients are small and the change in vorticity is then a second- or higher-order perturbation. The occurrence of a transverse velocity component may be explained in the following way. Although vorticity cannot be created, a density gradient can be produced by the non-uniform heat addition. The mass flow rate at a point $(+\infty, y)$ far downstream of the heated region differs from that at the corresponding point $(-\infty, y)$ far upstream and the streamline displacement to effect this change requires a transverse velocity perturbation.

It is thought that the study of a few simple cases of non-uniform heat addition, such as the examples given in this paper, may help to illustrate some of the main features of a diabatic flow. Although the two solutions given are for irrotational flows, they do show the presence of a local transverse velocity, an over-all density change, a density gradient across the duct which is retained far downstream and a streamline displacement. A more general form of the heat addition function will show all of these characteristics together with a first-order vorticity perturbation.

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